

TORSION OF A NON-CIRCULAR BAR

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The contribution deals with strain-stress analysis of torsion of a non-circular bar. Mathematical model is exactly derived and solutions are introduced and visualized for cases of triangular, rectangular and some other profiles.

Keywords: torsion of non-circular bar, Airy stress function, rectangular profile

1. Introduction

Analysis of properties, states and behavior of technical objects is an important task of Engineering Mechanics. Strain-stress analysis of solid bodies ranks to these problems. Continuum mechanics and especially elasticity theory provides tools for this analysis.

The strain-stress analysis passed through its development, analytic approaches predominating in the past are at present replaced by numerical tools as Finite Element Method, Finite Volume Method, Boundary Element Method etc. In comparison to the classical methods the numerical methods are universal in sense that their applicability is independent of geometry of the body, material characteristics, etc. This may indicate that the period of analytical elasticity ended. But it is not the case. There are reasons to keep on using both the analytical methods and the numerical methods in the strain-stress analysis.

The numerical methods (e.g. FEM) enable to compute numerical values of strain-stress state of the particular material in particular points under particular loads. On the other hand they provide no formulas which can be used for predicting the change of the values under changing the load, size, stiffness etc. The analytical methods yielding formulas enable this prediction. Unfortunately they can be applied only to special shapes of bodies and loads.

Also interpreting results obtained by numerical computations requires basic knowledge of mathematical elasticity, which causes troubles to many engineers in practice and leads to doubts whether the results obtained by numerical methods are correct. Analytical theory of elasticity can be a tool for verification of the numerical results.

This knowledge, hidden in formulas obtained by analytical methods, is a suitable tool for finding dangerous places of loaded bodies. For example torsion of a bar with cross-section that can be split into various rectangles. Analysis of shear stress of the particular rectangles enables to deduce the following results: in the case of open profile the maximal shear stress appears in rectangles with maximal thickness, in the case of closed profile it appears in the rectangles with minimal thickness. These results can be obtained also by numerical

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methods, but it requires computations of many variants of the profile.

We believe that the introduced facts sufficiently justify publishing a contribution dealing with an analytic method in the present period of numerical computations.

Torsion of the elastic bars is studied in several textbooks, see e.g. [3], [4], [2], [8], but the results are mostly introduced without proofs or circular cross-section only is considered, e.g. in monograph [1]. In this circular case the cross-sections remains planar, but in case of non-circular bar, the real cross-sections are deflected from the planar shape. The equation for a non-circular bar is derived correctly in [7], but no solutions for particular profiles are introduced.

The aim of this contribution is to fill in this gap: torsion of a bar with constant profile is analyzed using the Airy stress function. Complete exact derivation of the mathematical model is introduced. The second part contains explicit solutions for some non-circular profiles: starting with profiles given by polynomials. The case of rectangular profile is solved by means of Fourier series. The solutions (Airy stress function, modulus of shear stress and deflexion) are visualized in pictures by their level curves drawn by the system MAPLE. The case of hyperbolic section profile seems to be new.

Let us mention an older monograph [5] published in 1953 by Anselm Kovář in Czech. It starts with history of the torsion theory: the first solution of the non-circular case in 1836 by L. Navier assumed planar cross-sections which led to incorrect results: the stress attains its maximum in the most away points from the center of the cross-section. Using continuum mechanics B. Saint-Venant 1847 published the correct solution. The Kovář's monograph contains both technical and general mathematical torsion theory. It contains solution for several profiles: rectangle (by four methods), equilateral triangle, regular hexagon, octagon and other profiles, in [6] even the regular pentagon is solved; but the results are not obtained by the Airy stress method.

2. Theory

We shall consider an isotropic homogeneous long prismatic bar. According to the tradition in mechanics the axis of the bar coincides with x -axis, the cross-section denoted by Ω is a set in the y, z plane, see Fig. 1. The bar is fixed at $x = 0$ base, the opposite base $x = \ell$ is twisted by angle $\ell \alpha$. We adopt the following assumptions:

- the cross-sections in the y, z -plane rotates as a rigid body. In the case of a non-circular shape Ω , the cross-section is not planar, it is deflected in the x -direction,
- the deflection and the twist rate α is constant along the whole length of the bar. Thus the problem is reduced to a two-dimensional one.

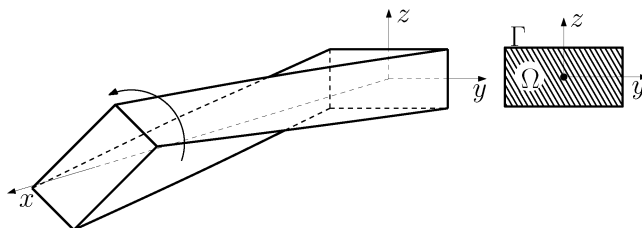


Fig.1: Orientation of the axis

2.1. Strain analysis

Let us assume the following geometrical behavior : According to the Saint-Venant hypothesis, the displacements u, v, w in directions x, y, z under these assumptions can be written in the form

$$\begin{aligned} u &= \alpha \varphi(y, z) , \\ v &= -\alpha x z , \\ w &= \alpha x y , \end{aligned} \quad (1)$$

where $\varphi(y, z)$ is an unknown function describing the deflection. It is supposed to be differentiable. Simple computation yields the corresponding strain (small deformation) tensor $e = \{e_{ij}\}$

$$\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x} = 0 , & e_{yy} &= \frac{\partial v}{\partial y} = 0 , & e_{zz} &= \frac{\partial w}{\partial z} = 0 , \\ e_{xy} &= \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2} \alpha \left(\frac{\partial \varphi}{\partial y} - z \right) , \\ e_{xz} &= \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \frac{1}{2} \alpha \left(\frac{\partial \varphi}{\partial z} + y \right) , \\ e_{yz} &= \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2} (-\alpha x + \alpha x) = 0 . \end{aligned} \quad (2)$$

Simple computation yields

$$\frac{\partial e_{xz}}{\partial y} - \frac{\partial e_{xy}}{\partial z} = \frac{\alpha}{2} \left(\frac{\partial^2 \varphi}{\partial y \partial z} + 1 - \frac{\partial^2 \varphi}{\partial z \partial y} + 1 \right) = \alpha . \quad (3)$$

2.2. Stress analysis

The Hooke's law of linear elasticity is written in the form:

$$\tau_{ij} = \lambda \delta_{ij} (e_{xx} + e_{yy} + e_{zz}) + 2 \mu e_{ij} , \quad (4)$$

where Kronecker's delta $\delta_{ij} = 1$ for $i = j$, otherwise $\delta_{ij} = 0$, and λ, μ are the Lamé constants. The sheer modulus μ is also often denoted by G . In our case the trace $e_{xx} + e_{yy} + e_{zz}$ equals to zero. Substituting (2) into (4) we obtain the components of the stress tensor $\tau = \{\tau_{ij}\}$

$$\begin{aligned} \tau_{xx} &= \tau_{yy} = \tau_{zz} = \tau_{yz} = 0 , \\ \tau_{xy} &= 2 \mu e_{xy} = \alpha \mu \left(\frac{\partial \varphi}{\partial y} - z \right) , \\ \tau_{xz} &= 2 \mu e_{xz} = \alpha \mu \left(\frac{\partial \varphi}{\partial z} + y \right) . \end{aligned} \quad (5)$$

The equilibrium equations $\sum_j \partial_j \tau_{ij} = f_i$ with zero forces f_i reduce to

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 , \quad \frac{\partial \tau_{xy}}{\partial x} = 0 , \quad \frac{\partial \tau_{xz}}{\partial x} = 0 . \quad (6)$$

For a simply connected domain Ω the equalities in (6) yield existence of a function $\Phi(y, z)$ independent of x such that the only nonzero stress components τ_{xy} and τ_{xz} given by

$$\tau_{xy} = \alpha \mu \frac{\partial \Phi(y, z)}{\partial z} , \quad \tau_{xz} = -\alpha \mu \frac{\partial \Phi(y, z)}{\partial y} \quad (7)$$

satisfy all the equilibrium equalities (6). Let us express the components e_{xy} and e_{xz} using (5) by means of Φ

$$e_{xy} = \frac{1}{2\mu} \tau_{xy} = \frac{\alpha}{2} \frac{\partial \Phi}{\partial z}, \quad e_{xz} = \frac{1}{2\mu} \tau_{xz} = -\frac{\alpha}{2} \frac{\partial \Phi}{\partial y} \quad (8)$$

and using (3) we obtain

$$\frac{\partial e_{xz}}{\partial y} - \frac{\partial e_{xy}}{\partial z} = -\frac{\alpha}{2} \left(\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right) = \alpha .$$

The last equality can be rewritten into an inhomogeneous second order partial differential equation

$$\Delta \Phi \equiv \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = -2. \quad (9)$$

2.3. Boundary condition

The equation has to be completed by boundary conditions. We assume that the boundary Γ is a piecewise differentiable simple curve and can be expressed by equations with parameter s being the arc length of the curve Γ . The curve is oriented such that the domain is on the left-hand side when the parametr s grows, see Fig. 2. Except for the ‘corners’ the unit outer vector $n = (n_y, n_z)$ normal to domain Ω exists on the boundary curve Γ .

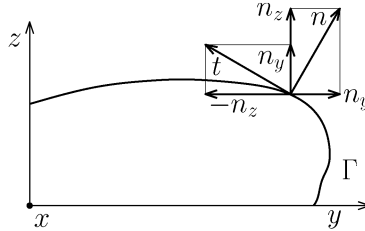


Fig.2: Unit vectors normal and tangent to Γ

Since zero surface forces are considered, on the boundary Γ the traction vector $T = (T_x, T_y, T_z)$ has zero components. Inserting τ_{xy}, τ_{xz} from (7) to equality $T_x = 0$ we obtain

$$T_x = \tau_{xy} n_y + \tau_{xz} n_z = \mu \alpha \left(\frac{\partial \Phi}{\partial z} n_y - \frac{\partial \Phi}{\partial y} n_z \right) = 0 . \quad (10)$$

But $(-n_z, n_y)$ is the tangent vector to boundary Γ of domain Ω , see Fig.2. Thus (10) implies that the tangent derivative of Φ equals to zero, and therefore Φ is constant along each component of the boundary Γ . For a profile with no holes the boundary Γ is connected and we can choose the constant to be zero. Thus we have arrived to the boundary value problem for the so-called Airy stress function $\Phi(y, z)$

$$\Delta \Phi \equiv \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = -2 \quad \text{in } \Omega ,$$

$$\Phi = 0 \quad \text{on } \Gamma .$$

If the profile Ω has holes (the so-called multi-connected domain Ω) then the boundary Γ is not connected. Besides the outer curve Γ_0 the boundary Γ consists of one or more

components Γ_i . Then the stress function can attain different values on each Γ_i , i.e. there exist constants c_i such that $\Phi = c_i$ on Γ_i .

2.4. Torque, section moment and stress

Let us compute the torque M of the twisted bar. It can be computed from the stress tensor τ as follows

$$M = \iint_{\Omega} (-\tau_{xy} z + \tau_{xz} y) dy dz .$$

Inserting from (7) we obtain

$$M = -\alpha \mu \iint_{\Omega} \left(\frac{\partial \Phi}{\partial z} z + \frac{\partial \Phi}{\partial y} y \right) dy dz .$$

We shall use the so-called integration by parts in the plane with z -derivative

$$\iint_{\Omega} \frac{\partial f}{\partial z} g dy dz = \int_{\Gamma} f g n_z ds - \iint_{\Omega} f \frac{\partial g}{\partial z} dy dz$$

and the similar formula for the y -derivative (both formulas follow from the Gauss-Ostrogradskii theorem). Since $\Phi = 0$ on the boundary Γ (in the case of the profile without holes) we get

$$M = 2 \alpha \mu \iint_{\Omega} \Phi dy dz .$$

We obtained dependence of the torque M on the twisting rate α

$$M = \alpha \mu J , \tag{12}$$

where the moment of the cross-section J is given by

$$J = 2 \iint_{\Omega} \Phi(y, z) dy dz . \tag{13}$$

In case of a profile $\Omega = \Omega_0 - \cup_i \overline{\Omega_i}$ with holes Ω_i putting $\Phi = 0$ on the outer boundary Γ_0 and $\Phi = c_i$ on the boundary curves Γ_i of holes then

$$J = 2 \iint_{\Omega} \Phi dy dz + 2 \sum_i c_i |\Omega_i| , \tag{14}$$

where $|\Omega_i|$ is volume of the hole Ω_i .

How to find maximum stress? In our case the modulus of the stress force $T(n) = \tau_{xy} n_y + \tau_{xz} n_z$ is $|T| = [\tau_{xy}^2 + \tau_{xz}^2]^{1/2}$. If the cross-section Ω is a convex set and function Φ is zero on Γ , then the function Φ is strictly concave and its y (or z) derivative is decreasing in y (or z) direction. Thus the derivatives cannot attain their extremes inside Ω . Since the tangent derivative of Φ is zero at Γ , the modulus equals the normal derivative of Φ . Thus the maximum can be found on the boundary Γ , at the boundary point which is the nearest

to the point of maximum value of Φ , since there is the biggest slope of Φ . The maximum value of the stress $|T|_{\max}$ which is important in engineering practice is often expressed in the form $|T|_{\max} = M/W$, where M is the torque and W is a quantity called the twist section modulus. It is defined

$$W = \frac{M}{|T|_{\max}} = \frac{\alpha \mu J}{|T|_{\max}}. \quad (15)$$

and for particular shapes is expressed by means of section dimensions and a shape constant.

2.5. Summary of the results

Let us summarize the results. In case of the profile without holes we compute the Airy stress function $\Phi(y, z)$ as the solution of the boundary value problem (11). Then (13) yields the torsion constant J and from (12) we get dependence of the twist rate α on the torque M . The non-zero components τ_{xy} , τ_{xz} of the stress tensor are given by (7).

To obtain the displacements we need to compute the deflexion function φ . Combining (8) and (5) we obtain

$$\frac{\partial \varphi}{\partial y} = \frac{\partial \Phi}{\partial z} + z, \quad \frac{\partial \varphi}{\partial z} = -\frac{\partial \Phi}{\partial y} - y. \quad (16)$$

It is the problem of finding a potential $\varphi(y, z)$ from its differential

$$d\varphi = f dy + g dz.$$

Easy calculation verifies that due to equation (9) the compatibility condition $\partial f/\partial z - \partial g/\partial y = 0$ is satisfied, thus the potential φ exists up to an additive constant. To obtain unique deflection function φ we add zero integral mean condition

$$\iint_{\Omega} \varphi(y, z) = 0. \quad (17)$$

Then the displacements u , v and w are given by (1).

Let us remark the dimensions of the quantities. The displacements u , v , w and deflection function φ are in meters [m], strain tensor e is dimensionless, twist rate α is in $[\text{m}^{-1}]$, Airy stress function Φ is in $[\text{m}^2]$, moment of the cross-section J in $[\text{m}^4]$, the twist section modulus W in $[\text{m}^3]$, the shear modulus μ , the stress tensor τ , the stress vector T in $[\text{N m}^{-2}]$ and the torque M in $[\text{N m}]$.

3. Examples

The crucial point of the computation is solving the boundary value problem (11). We start with polynomial stress functions. Let a polynomial $P(y, z)$ satisfy

$$\Delta P \equiv \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = -2. \quad (18)$$

If the contour line $\Gamma = \{(y, z) \in \mathbb{R}^2 \mid \Phi(y, z) = 0\}$ bounds a bounded non-empty domain Ω , then we obtained the solution of the boundary value problem (11) on a cross-section Ω . Since in case of convex Ω the polynomial P is strictly concave, also the problem is solved also for cross-sections Ω given by $P(y, z) > c$ for positive constants $c > 0$.

Considering general second order polynomial

$$P_2(y, z) = a_{20} y^2 + a_{11} yz + a_{02} z^2 + a_{10} y + a_{01} z + a_{00}$$

we obtain $\Delta P_2(y, z) = 2a_{20} + 2a_{02}$, which yields the condition

$$a_{20} + a_{02} = -1. \quad (19)$$

Thus any polynomial P_2 satisfying the condition (19) solves the equation (18). In this case the only bounded cross-sections are ellipses including the circle. The case of ellipse cross-sections will be solved in Section 3.1.

Let us consider a third order polynomial

$$P_3(y, z) = a_{30} y^3 + a_{21} y^2 z + a_{12} y z^2 + a_{03} z^3 + P_2(y, z).$$

Simple computation yields

$$\Delta P_3 = 6a_{30} y + 2a_{21} z + 2a_{12} y + 6a_{03} z + 2a_{20} + 2a_{02}.$$

To obtain a solution of (18) we complete (19) by two additional conditions

$$3a_{30} + a_{12} = 0, \quad a_{21} + 3a_{03} = 0.$$

Thus for any $c_1, c_2 \in \mathbb{R}$ and any polynomial P_2 satisfying (19) the polynomial

$$P_3(y, z) = c_1 y (y^2 - 3z^2) + c_2 z (3y^2 - z^2) + P_2(y, z)$$

is a solution to (18). The only bounded cross-sections determined by third order polynomials are hyperbolic segments studied in Section 3.4, and their limit cases equilateral triangles studied in Section 3.3. Besides these cross-sections Ω also their subsets Ω_c given by $\{[y, z] \mid P_3(y, z) > c \text{ for } c > 0\}$ are solved.

In the case of a fourth order polynomial

$$P_4(y, z) = a_{40} y^4 + a_{31} y^3 z + a_{22} y^2 z^2 + a_{13} y z^3 + a_{04} z^4 + P_3(y, z)$$

a similar computation yields additional conditions

$$6a_{40} + a_{22} = 0, \quad a_{31} + a_{13} = 0, \quad a_{22} + 6a_{04} = 0$$

to the previous ones. Thus for any $c_1, c_2, c_3, c_4 \in \mathbb{R}$ and any second order polynomial $P_2(y, z)$ satisfying (19) the fourth order polynomial

$$P_4(x, y) = c_1 (y^4 - 6y^2 z^2 + z^4) + c_2 y z (y^2 - z^2) + c_3 y (y^2 - 3z^2) + c_4 z (3y^2 - z^2) + P_2(y, z)$$

is a solution to the problem (18). We only need to check whether the polynomial yields a bounded cross-section. In this way we can proceed to polynomials of higher orders. On the other hand the question is whether the obtained cross-sections Ω are bounded and ‘useful’ in engineering practice.

3.1. Elliptic profile

The only bounded Ω given by second order polynomials are ellipses. Let us consider the ellipse Γ with semi axes $a \geq b > 0$ in the standard position

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 .$$

The corresponding stress function Φ

$$\Phi(y, z) = k \left(\frac{y^2}{a^2} + \frac{z^2}{b^2} - 1 \right)$$

clearly satisfies $\Phi = 0$ on Γ and condition $\Delta\Phi = -2$ yields value of the constant $k = -(a^{-2} + b^{-2})^{-1}$. Thus the solution Φ of the problem (11) is

$$\Phi(y, z) = \frac{a^2 b^2}{a^2 + b^2} \left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2} \right) . \quad (20)$$

The obtained Airy stress function Φ is plotted by its level curves on Fig. 3a. The moment J of the cross-section can be computed using (13). Using the elliptic coordinates $y = a \rho \cos \varphi$, $z = b \rho \sin \varphi$ with Jacobian $a b \rho$ we obtain

$$J = 2 \iint_{\Omega} \Phi(y, z) dy dz = 4 \pi \frac{a^3 b^3}{a^2 + b^2} \int_0^1 (1 - \rho^2) \rho d\rho = \pi \frac{a^3 b^3}{a^2 + b^2} .$$

In the case of circular cross section Ω with $a = b = R$ we obtain $J = \pi R^4/2$, see e.g. [1], page 251.

From (5) we can see that the only non-zero components of the stress tensor are

$$\tau_{xy}(y, z) = -2 \alpha \mu \frac{a^2 z}{a^2 + b^2} , \quad \tau_{xz}(y, z) = 2 \alpha \mu \frac{b^2 y}{a^2 + b^2} .$$

The modulus of the shear stress

$$|T| = \sqrt{\tau_{xy}^2 + \tau_{xz}^2} = 2 \alpha \mu \frac{\sqrt{a^4 z^2 + b^4 y^2}}{a^2 + b^2}$$

is plotted on the figure by its level curves on Fig. 3b. It attains its maximum $|T|_{\max} = 2 \alpha \mu a^2 b / (a^2 + b^2)$ in points $[0, \pm b]$. The twist section modulus is $W = \pi a b^2 / 2$.

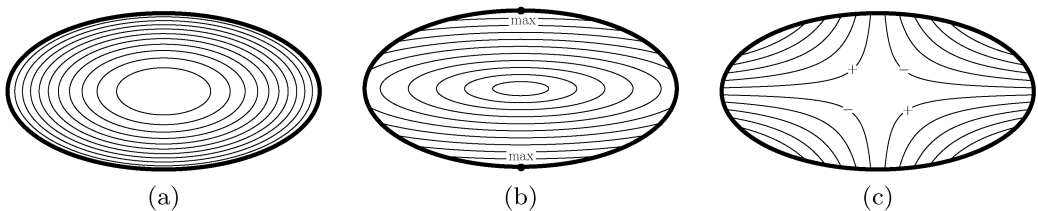


Fig.3: Elliptic cross-section with $\mu = 1$, $\alpha = 1$, $a = 1$ and $b = 1/2$: (a) stress function Φ , (b) modulus of the shear stress T , (c) deflection function φ

Concerning the deflection φ , in our case the equations (16) read

$$\frac{\partial \varphi}{\partial y} = 2z \frac{a^2}{a^2 + b^2} + z, \quad \frac{\partial \varphi}{\partial z} = -2y \frac{b^2}{a^2 + b^2} - y.$$

Standard computation with (17) yields the deflection function φ

$$\varphi(y, z) = -\frac{a^2 - b^2}{a^2 + b^2} y z,$$

which is also plotted on Fig. 3c.

3.2. Ring profile

Circular tubes are frequent elements used in engineering practice. Torsion of tubes can be solved by means of the stress function from the previous section. Let $\Omega = \{[y, z] \mid r^2 < y^2 + z^2 < R^2\}$ be a ring with outer R and inner r radius; $R > r > 0$. Putting $a = b = R$ in (20) we obtain the Airy stress function

$$\Phi(y, z) = \frac{1}{2} (R^2 - y^2 - z^2).$$

It takes value $\Phi = 0$ on the outer boundary $\Gamma_0 = \{[y, z] \mid y^2 + z^2 = R^2\}$ and value $\Phi = (R^2 - r^2)/2$ on the inner boundary $\Gamma_1 = \{[y, z] \mid y^2 + z^2 = r^2\}$. The moment J of the cross-section Ω can be computed using (14)

$$J = 2 \iint_{\Omega} \frac{1}{2} (R^2 - y^2 - z^2) dy dz + \frac{1}{2} (R^2 - r^2) \pi r^2 = \frac{\pi}{2} (R^4 - r^4).$$

The shear stress and the deflection follow directly from those of the elliptic case

$$\tau_{xy}(y, z) = -\alpha \mu z, \quad \tau_{xz}(y, z) = \alpha \mu y, \quad |T|_{\max} = \alpha \mu R, \quad \varphi(y, z) = 0.$$

The result functions are not interesting, thus they are not plotted.

3.3. Triangular profile

Third order polynomials enable to solve the case of a triangle profile. Let us consider a general triangle Ω with one side being parallel to the axis y , i.e. a subset of the line $z = -c$. Let the other sides lie on the lines $z = a_1 y + b_1$ and $z = a_2 y + b_2$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and $c > 0$. Then the level curve $P_3(y, z) = 0$ of the polynomial

$$P_3(y, z) = k (a_1 y + b_1 - z) (a_2 y + b_2 - z) (z + c)$$

with $k \neq 0$ bounds the triangle Ω . Simple calculation yields

$$\Delta P_3(y, z) = -2k (a_1 + a_2) y + 2k (a_1 a_2 + 3) z + 2k (a_1 a_2 c - b_1 - b_2 + c).$$

The right-hand side is constant if the coefficients by terms y and z are zero, i.e. if $a_1 + a_2 = 0$ and $a_1 a_2 + 3 = 0$, which implies $a_1 = -a_2 = \pm\sqrt{3}$. Thus the triangle Ω must be equilateral.

Choosing $b_1 = b_2 = b$ and $b = 2c$ the triangle has vertices $[-\sqrt{3}c, -c]$, $[\sqrt{3}c, -c]$, $[0, 2c]$ with its center of gravity in $[0, 0]$.

The polynomial P_3 is a solution to the problem (11) if the right-hand constant equals to -2 , i.e. $-k(a_1 a_2 c - b_1 - b_2 + c) = k 6c = 1$ which yields $k = (6c)^{-1}$. We obtained the Airy stress function

$$\Phi(y, z) = \frac{1}{6c} (\sqrt{3}y + 2c - z) (-\sqrt{3}y + 2c - z) (z + c). \quad (21)$$

The moment J of the cross-section can be computed using (13):

$$J = 2 \frac{1}{6c} \int_{-c}^{2c} \left(\int_{(z-2c)/\sqrt{3}}^{(2c-z)/\sqrt{3}} \Phi(y, z) dy \right) dz = \frac{9\sqrt{3}}{5} c^4.$$

The only non-zero components of the stress tensor are

$$\tau_{xy} = -\alpha \mu \frac{y^2 - z^2 + 2cz}{2c}, \quad \tau_{xz} = \alpha \mu \frac{y(z+c)}{c}.$$

The modulus of the shear stress

$$|\tau| = \sqrt{\tau_{xy}^2 + \tau_{xz}^2} = \alpha \mu \frac{1}{2c} \sqrt{(y^2 - z^2 + 2cz)^2 + 4y^2(z+c)^2}$$

is plotted on the figure 4b, maximum values $|T|_{\max} = \alpha \mu 3c/2$ are attained in the center points of the sides, e.g. in $[0, -c]$. The twist section modulus is $W = 6\sqrt{3}c^3/5$. Concerning the deflection φ , in our case the equations (16) read

$$\frac{\partial \varphi}{\partial y} = \frac{z^2 - 2zc - y^2}{2c} + z, \quad \frac{\partial \varphi}{\partial z} = \frac{(z+c)y}{c} - y.$$

Standard computation with (17) yields the deflection function φ

$$\varphi(y, z) = \frac{1}{6c} y (3z^2 - y^2),$$

which is also plotted on Fig. 4c.

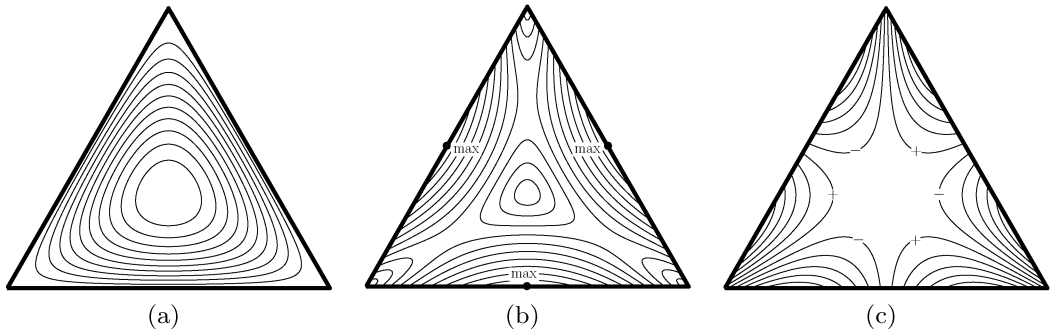


Fig.4: Triangular cross-section with $\mu = 1$, $\alpha = 1$ and $c = 1$: (a) stress function Φ , (b) modulus of the shear stress τ , the maximum values of $|T|$ occur in the middle of the sides, (c) deflection of the cross-section

3.4. Hyperbolic segment profile

The third order polynomials can solve also a special case of hyperbolic segment, i.e. a bounded domain Ω limited by a line $y = d > 0$ parallel to z axis and a hyperbola $(y + c)^2/a^2 - z^2/b^2 = 1$ in the y -shifted position with the center $(-c, 0)$. The level curve $P_3(x, y) = 0$ of the polynomial

$$P_3(y, z) = k \left(\frac{(y + c)^2}{a^2} - \frac{z^2}{b^2} - 1 \right) (y - d)$$

bounds the domain Ω under consideration. Simple computation yields

$$\Delta\Phi(y, z) = 2k \frac{(3b^2 - a^2)y + (a^2 - b^2)d + 2cb^2}{a^2b^2}.$$

The right-hand side is constant if $3b^2 - a^2 = 0$, i.e. $a = \sqrt{3}b$. Let us remark that in this case hyperbola asymptotes along with line $y = d$ bound an equilateral triangle. Further the right-hand side equals to -2 , if $k = -a^2/(2(d + c))$ and the Airy stress function is

$$\Phi(y, z) = \frac{(a^2 - (y + c)^2 + 3z^2)(y - d)}{2(c + d)}. \quad (22)$$

With a convenient shift c the cross section Ω has its center of gravity in the origin. Since analytic expression of such constant c is complicated (in practise it can be obtained only numerically) we will leave it undetermined.

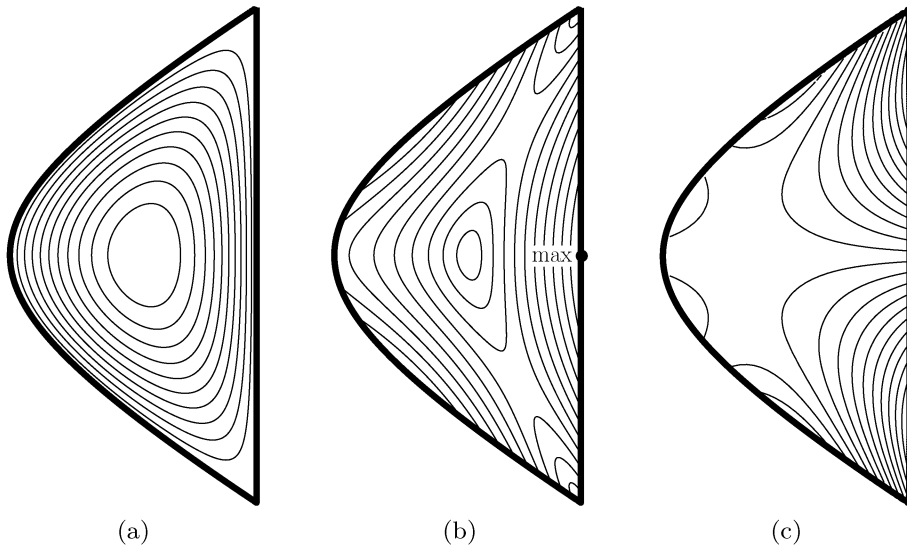


Fig.5: Hyperbolic cross-section with $\mu = 1$, $\alpha = 1$, $a = 1$, $d = 2$: (a) stress function Φ , (b) modulus of the shear stress τ , (c) deflection of the cross-section

Components of the stress tensor τ are

$$\tau_{xy} = \alpha \mu \frac{3(y - d)z}{c}, \quad \tau_{xz} = \alpha \mu \frac{3(y^2 - z^2) + 2(2c - d)y - a^2 - 2cd + c^2}{2(c + d)},$$

the others are equal to zero. The modulus of the shear stress $|T|$ is plotted in the figure 5b. Its maximum value is either $|T|_{\max} = \alpha \mu ((c+d)^2 - a^2)/(2(c+d))$ in point $[d, 0]$ if $c+d \geq \sqrt{2}a$ or $|T|_{\max} = \alpha \mu a^2/(2(c+d))$ in point $[-c, 0]$ if $c+d \leq \sqrt{2}a$.

Concerning the deflection φ , in our case the equations (16) read

$$\frac{\partial \varphi}{\partial y} = \frac{(3y - 2d + c)z}{c + d}, \quad \frac{\partial \varphi}{\partial z} = \frac{3(y^2 - z^2) + 2(c - 2d)y - a^2 - 2cd + c^2}{2(c + d)}.$$

It is not difficult to compute φ

$$\varphi(y, z) = \frac{3y^2z - z^3 + 2(c - 2d)yz - (a^2 + 2cd - c^2)z}{2(c + d)},$$

where the integration constant is zero, since φ is odd in z and Ω is symmetric in z . Level curves of φ are plotted on the figure 5c.

3.5. Non-polynomial stress functions

In general case the cross-section cannot be expressed as a level-curve of a polynomial satisfying the equation (18). Even such case as rectangular parallelepiped cannot be solved by polynomial stress function. Indeed, a centered rectangle with sides a and b is a level curve of a polynomial

$$P_4(y, z) = k \left(y - \frac{a}{2}\right) \left(y + \frac{a}{2}\right) \left(z - \frac{b}{2}\right) \left(z + \frac{b}{2}\right),$$

but $\Delta P_4(y, z) = k[2(y^2 + z^2) + (a^2 + b^2)/2]$, which cannot equal to a nonzero constant. Nevertheless, the problem can be solved by means of Fourier series.

3.6. Rectangular profile

Rectangular profiles are very important in engineering practice. We shall consider a rectangle $\Omega = (-a/2, a/2) \times (-b/2, b/2)$ with $a \geq b > 0$. The Airy stress function will be looked for in form of a double sum of functions having zero values at the boundary Γ . To this purpose we choose products of $\cos(k\pi y/a)$ and $\cos(l\pi z/b)$ with positive odd integers k, l since they are zero at Γ and have ‘good’ derivatives. Let us denote the set of positive odd integers by $\mathbb{N}_{\text{odd}} = \{1, 3, 5, 7, \dots\}$ and put

$$\Phi(y, z) = \sum_{k \in \mathbb{N}_{\text{odd}}} \sum_{l \in \mathbb{N}_{\text{odd}}} c_{kl} \cos \frac{k\pi y}{a} \cos \frac{l\pi z}{b}.$$

with some coefficients c_{kl} . The double sum is supposed to satisfy the equation (9). Inserting Φ into the left hand side of (9) we obtain

$$\Delta \Phi(y, z) \equiv \sum_{k \in \mathbb{N}_{\text{odd}}} \sum_{l \in \mathbb{N}_{\text{odd}}} -c_{kl} \left[\frac{k^2 \pi^2}{a^2} + \frac{l^2 \pi^2}{b^2} \right] \cos \frac{k\pi y}{a} \cos \frac{l\pi z}{b} = -2. \quad (23)$$

To find the coefficients c_{kl} we express -2 by means of the series of the same type. Let us expand the even function $f(y) = 1$ on $(-a/2, a/2)$ into a cosine series

$$\sum_{k \in \mathbb{N}_{\text{odd}}} c_k \cos \frac{k\pi y}{a}.$$

Since $\{\cos(k\pi y/a), k \in \mathbb{N}_{\text{odd}}\}$ form a complete orthogonal sequence in the Hilbert space of even square integrable functions on $(-a/2, a/2)$, the coefficients c_k with odd k are given by

$$c_k = \frac{2}{a} \int_0^{a/2} f(y) \cos \frac{k\pi y}{a} dy = \frac{4}{a} \left[\frac{a}{k\pi} \sin \frac{k\pi y}{a} \right]_0^{a/2} = (-1)^{\frac{k-1}{2}} \frac{4}{k\pi}.$$

In the same way we expand the function $g(z) = 1$ on $(-b/2, b/2)$. Thus we obtained

$$\sum_{k \in \mathbb{N}_{\text{odd}}} (-1)^{\frac{k-1}{2}} \frac{4}{k\pi} \cos \frac{k\pi y}{a} = 1, \quad \sum_{l \in \mathbb{N}_{\text{odd}}} (-1)^{\frac{l-1}{2}} \frac{4}{l\pi} \cos \frac{l\pi z}{b} = 1.$$

Multiplying product of both series with constant -2 we obtain equality

$$\sum_{k \in \mathbb{N}_{\text{odd}}} \sum_{l \in \mathbb{N}_{\text{odd}}} -2(-1)^{\frac{k+l}{2}-1} \frac{4^2}{kl\pi^2} \cos \frac{k\pi y}{a} \cos \frac{l\pi z}{b} = -2. \quad (24)$$

Since both series in (23) and (24) are of the same type and have the same sum, the corresponding coefficient must be equal. Comparing both series we obtain formula for the coefficients c_{kl} :

$$c_{kl} = 2 \frac{4^2}{kl\pi^2} (-1)^{\frac{k+l-2}{2}} \left[\frac{k^2\pi^2}{a^2} + \frac{l^2\pi^2}{b^2} \right]^{-1} = \frac{2^5 a^2 b^2}{\pi^4} \frac{(-1)^{\frac{k+l}{2}-1}}{kl(k^2 b^2 + l^2 a^2)}.$$

Thus the Airy stress function equals to

$$\Phi(y, z) = \frac{2^5 a^2 b^2}{\pi^4} \sum_{k \in \mathbb{N}_{\text{odd}}} \sum_{l \in \mathbb{N}_{\text{odd}}} \frac{(-1)^{\frac{k+l}{2}-1}}{kl(k^2 b^2 + l^2 a^2)} \cos \frac{k\pi y}{a} \cos \frac{l\pi z}{b}. \quad (25)$$

We obtained the result in the form of an infinite series. Due to estimate

$$\left| c_{kl} \cos \frac{k\pi y}{a} \cos \frac{l\pi z}{b} \right| \leq |c_{kl}| \leq \text{const.} \frac{1}{k^2 l^2},$$

the series converge uniformly, since $\sum_{k \in \mathbb{N}_{\text{odd}}} \sum_{l \in \mathbb{N}_{\text{odd}}} 1/(k^2 l^2)$ equals to product of two series $(\sum_{k \in \mathbb{N}_{\text{odd}}} 1/k^2)(\sum_{l \in \mathbb{N}_{\text{odd}}} 1/l^2)$ which both have finite sum. Let us compute the moment J . Since

$$\int_{-a/2}^{a/2} \cos \left(\frac{k\pi y}{a} \right) dy = \frac{2a}{k\pi} (-1)^{\frac{k-1}{2}}, \quad \int_{-b/2}^{b/2} \cos \left(\frac{l\pi z}{b} \right) dz = \frac{2b}{l\pi} (-1)^{\frac{l-1}{2}}$$

the formula (13) yields

$$J = 2^8 \frac{a^3 b^3}{\pi^6} \sum_{k \in \mathbb{N}_{\text{odd}}} \sum_{l \in \mathbb{N}_{\text{odd}}} \frac{1}{k^2 l^2 (k^2 b^2 + l^2 a^2)}.$$

Let us denote the ratio $r = a/b$. Using a function $K_1(r)$ the relation can be rewritten to

$$J = K_1 \left(\frac{a}{b} \right) a b^3,$$

r	1	1.5	2	3	4	5	6	8	10	∞
$K_1(r)$	0.141	0.196	0.229	0.263	0.281	0.291	0.298	0.307	0.312	1/3
$K_2(r)$	0.208	0.231	0.246	0.267	0.282	0.292	0.299	0.307	0.313	1/3

Tab.1: Table with the numerical values of $K_1(r)$ and $K_2(r)$

where the dimensionless function $K_1(r)$ depending on the ratio r is given by

$$K_1(r) = \frac{2^8}{\pi^6} \sum_{k \in \mathbb{N}_{\text{odd}}} \sum_{l \in \mathbb{N}_{\text{odd}}} \frac{r^2}{k^2 l^2 (k^2 + r^2 l^2)}.$$

The values of $K_1(r)$ for some ratios r are in Table 1.

Relation (7) yields the nonzero components of the stress tensor

$$\begin{aligned} \tau_{xy}(y, z) &= \alpha \mu \frac{2^5 a^2 b}{\pi^3} \sum_{k \in \mathbb{N}_{\text{odd}}} \sum_{l \in \mathbb{N}_{\text{odd}}} \frac{(-1)^{\frac{k+l}{2}}}{k (k^2 b^2 + l^2 a^2)} \cos \frac{k \pi y}{a} \sin \frac{l \pi z}{b}, \\ \tau_{xz}(y, z) &= -\alpha \mu \frac{2^5 a b^2}{\pi^3} \sum_{k \in \mathbb{N}_{\text{odd}}} \sum_{l \in \mathbb{N}_{\text{odd}}} \frac{(-1)^{\frac{k+l}{2}}}{l (k^2 b^2 + l^2 a^2)} \sin \frac{k \pi y}{a} \cos \frac{l \pi z}{b}. \end{aligned}$$

According to the remark in the end of Section 2, the maximum of the stress is in the middle point $[0, \pm b/2]$ of the longer side of the rectangle. From (7) simple computation yields

$$|T|_{\text{max}} = \alpha \mu \frac{2^5 a^2}{\pi^3 b} \sum_{k \in \mathbb{N}_{\text{odd}}} \sum_{l \in \mathbb{N}_{\text{odd}}} \frac{(-1)^{\frac{k-1}{2}}}{k (k^2 + r^2 l^2)}.$$

Using a function $K_2(r)$ the relation can be rewritten to

$$|T|_{\text{max}} = \alpha \mu a b^2 K_2 \left(\frac{a}{b} \right),$$

where the dimensionless function $K_2(r)$ depending on the ratio $r = a/b$ and $K_1(r)$ is given by

$$K_2(r) = K_1(r) \frac{\pi^3}{2^5 r^2} \left[\sum_{k \in \mathbb{N}_{\text{odd}}} \sum_{l \in \mathbb{N}_{\text{odd}}} \frac{(-1)^{\frac{k-1}{2}}}{k (k^2 + r^2 l^2)} \right]^{-1}.$$

Some numerical values of $K_2(r)$ are listed in the Table 1. In mechanics the $|T|_{\text{max}}$ is often expressed in the form $|T|_{\text{max}} = M/W$, where the section twisting moment W is given by $W = a b^2 K_2(a/b)$.

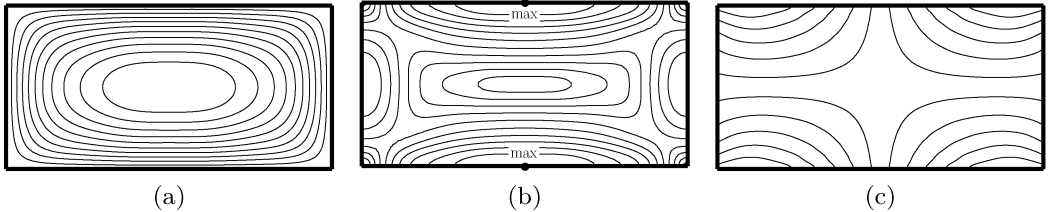


Fig.6: Results for a rectangular profile with $\mu = 1$, $\alpha = 1$, $a = 2$ and $b = 1$: (a) stress function Φ , (b) modulus of the shear stress τ ; the maximum values of τ occur in the middle of longer sides, (c) deflection of the cross-section

The deflection function φ is given by its differential. Using (16) we obtain

$$\varphi(y, z) = \frac{2^5 a^3 b}{\pi^4} \sum_{k \in \mathbb{N}_{\text{odd}}} \sum_{l \in \mathbb{N}_{\text{odd}}} \frac{(-1)^{\frac{k+l}{2}}}{k^2 (k^2 b^2 + l^2 a^2)} \sin \frac{k \pi y}{a} \sin \frac{l \pi z}{b} + y z,$$

its values are also plotted in the Figure 6c.

In case of more complicated profiles the analytic solution does not exist but the stress function can be found numerically, e.g. by the finite element method.

4. Conclusion

In the first part the mathematical model of torsion of a non-circular prismatic bar was built under the Saint-Venant hypothesis (1). All results were derived from well-known mechanical laws as equilibrium equations (6) and Hooke's law (4). The Section 3 provides several examples. Exact solution is given for some profiles with polynomial boundary, the sufficient conditions are derived. Classical examples like elliptic and triangular profiles are included. In the end the most important case is studied: the solution of rectangular profile is solved in the form of infinite series. The numerical values in Table 1 coincide with those in [8], [5].

Let us mention the influence of choice of the coordinate origin. Shifting Ω by a vector $s = [y_s, z_s]$ to $\Omega_s = \{[y + y_s, z + z_s] \mid [y, z] \in \Omega\}$ we obtain the boundary value problem (11) on Ω_s . Its solution, the Airy stress function Φ_s is shifted function Φ , namely $\Phi_s(y, z) = \Phi(y - y_s, z - z_s)$. Thus its shape is unchanged. Since the stress is derived from the Airy stress function, also the stress tensor τ and stress vector T are shifted without change of the shape, also the maximum stress is conserved.

The situation differs for the deflection function φ . It is a solution of the equation (16) which contains coordinates y and z on the right hand side. Thus the deflection function $\varphi(y, z)$ depends on the shift, its shape changes with the shift s . Which deflection is correct? The model is based on the Saint-Venant hypothesis (1) which describes the torsion around the x axis, i.e. around the origin $[y, z] = [0, 0]$. If it is the center of gravity, then the deformation given by (1) is acceptable. If the torsion axis (point $[0, 0]$) is not in the center of gravity of the cross-section Ω , then according to the hypotheses (1) the centers of gravity of the twisted bar lie on a helix which contradicts the behavior of the real material. Indeed, in this case the momentum equilibria condition are not satisfied. In all examples the center of rotation coincides with center of gravity of the cross-section, in the hyperbolic section case the shift parameter c is not expressed explicitly.

Most of the results are published in many textbooks of mechanics but usually without proofs or only for circular cylinders and accompanied by a few unsolved examples. In the present paper the mathematical model is derived in details and there are several fully solved examples including the rectangle. The example of hyperbolic cross-section seems to be new.

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